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VIBRATION ANALYSIS OF SKEW FIBRE-REINFORCED COMPOSITE LAMINATED PLATES

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The vibration of skew laminated composite plates with simply supported and clamped edges is studied. The skew plate is mapped into a unit square by linear transformation. Orthogonal polynomials are used with the Ritz method to determine the natural frequencies. The effects of skew angle and lamination scheme on natural frequencies are studied. Results are tabulated for different lamination schemes and compared to available studies in the literature.

1. INTRODUCTION

The use of skew laminated composite plates has increased in many applications. As a result, there is a significant number of studies on vibration of thin, skew isotropic plates (see, for example, references [1–16]). On the other hand, in literature there is a relatively limited number of studies on laminated composite skew plates. Krishnan and Desphande [17] studied thin cantilevered isotropic skew plates, laminae and laminates using discrete Kirchoff theory. Hosokawa *et al.* [18] analyzed free vibrations of a fully clamped symmetrically laminated skew plate, using the Green function approach. They studied the effects of the skew angle and the fibre orientation angle on natural frequencies and mode shapes. Han and Dickinson [19] extended the Ritz approach to symmetrically laminated, composite, skew plates. They illustrated the influence of different lamination lay-ups, skew angles and edge conditions on the natural frequencies and nodal patterns of a selection of plates. Wang [20] presented a B-spline Rayleigh–Ritz method for free vibration analysis of thin skew fibre-reinforced composite laminates. He obtained non-dimensional frequency parameters for arbitrary lay-ups, various skew angles and boundary conditions.

In the present study, a skew plate is mapped into a unit square, and the Ritz method is used together with two-dimensional orthogonal polynomials to study the free vibration analysis of thin skew laminated composite plates. To prove the validity of the approach, numerical results are compared to the results available in the literature.

2. SOLUTION

Figure 1 shows the geometry of a skew lamina and the fibre orientation. Each lamina is assumed to have the same material density, ρ, per unit area and the same thickness, h. The laminate is made up of a number of laminae, each consisting of unidirectional fibre-reinforced composite material. The skew angle α is measured as shown in Figure 1. The fibre angle, θ, is measured from the x-axis in the counterclockwise direction.

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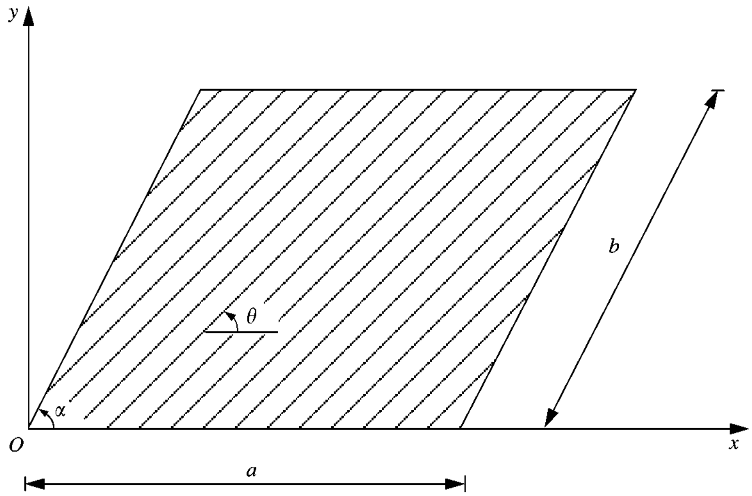

Figure 1. The geometry of a skew fibre-reinforced laminate.

The Ritz method is used to find an approximate solution for free vibration of skew laminated composite plates. The strain energy and the kinetic energy are given by

$$V = \frac{1}{2}\iint_R \left\{ D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right.$$

$$\left. + 4D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y} + 4D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial y} + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right\} \mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

$$T = \frac{1}{2}\rho_0 h \iint_R \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y. \qquad (2)$$

D_{ij}'s are the laminate stiffnesses given in Appendix A, ρ_0 is the mass per unit volume. Since the system is assumed to be conservative, the total energy obtained by adding equations (1) and (2) is constant. This displacement function is assumed to have the following form:

$$w(x, y, t) = W(x, y)\mathrm{e}^{\mathrm{j}\omega t}, \qquad (3)$$

where ω is the natural frequency and W the midsurface displacement in the z direction. Substituting equation (3) into equations (1) and (2), the following is obtained:

$$\frac{\mathrm{e}^{2\mathrm{j}\omega t}}{2}\left[\frac{h^3 E_{22}}{a^4}\iint_R \left\{ D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right) + 4D_{16}\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial x \partial y}\right)\right.\right.$$

$$\left.\left. + 4D_{26}\left(\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x \partial y}\right) + 4D_{66}\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 \right\} \mathrm{d}x\,\mathrm{d}y - \omega^2 \rho_0 h \iint_R W^2\,\mathrm{d}x\,\mathrm{d}y \right] = constant, \qquad (4)$$

where E_{22} is the modulus of a lamina transverse to the direction of fibres. The solution of the problem reduces to finding the minimum of the total energy given by equation (4).

2.1. MAPPING OF THE SKEW DOMAIN

Before assuming a proper function for the midplane displacement $W(x, y)$, the skew plate is mapped into a unit square. A similar mapping is used by Liew *et al.* [21] for the mapping of quadrilateral plates into a unit square. Figure 2(a) shows the domain of the skew plate R. The mapping of the skew plate into a unit square is performed as follows:

$$x = a\xi + b\cos\alpha\eta, \quad y = b\sin\alpha\eta, \tag{5}$$

where ξ and η are new co-ordinate axes as shown in Figure 2(b). If equation (5) is substituted into equation (4) and the derivatives evaluated accordingly, an energy equation similar to equation (4) in terms of ξ and η is obtained. To apply the Ritz method a displacement function for the midplane, $\varpi(\xi, \eta)$, which is made up of orthogonal polynomials will be used in the mapped domain.

Linearly independent set of polynomials satisfying essential boundary conditions can be defined as follows:

$$F_i(\xi, \eta) = f(\xi, \eta) f_i(\xi, \eta), \quad i = 1, 2, \ldots, \tag{6}$$

where $f_i(\xi, \eta)$ take the $\xi^{m_i}\eta^{n_i}$ form. m_i and n_i are non-negative integers and their choice depends upon the mode shapes. For modes symmetric about the ξ-axis and antisymmetric about the η-axis, m_i are odd and n_i are even. $f(\xi, \eta)$ satisfy the boundary conditions and have the following form:

$$f(\xi, \eta) = \xi^p (1 - \xi)^q \eta^r (1 - \eta)^s. \tag{7}$$

The parameters p, q, r and s take values according to the types of boundaries at the sides. The parameters will be 0, 1 or 2 as the sides are free, simply supported or clamped, respectively. If the $\xi = 0$ side is simply supported, then $p = 1$. The orthogonal polynomials, ϕ_k, are generated using the Gram–Schmidt orthogonalization procedure [13]. Generation of orthogonal polynomials is also proposed by Bhat [22, 23].

$$\phi_1(\xi, \eta) = f(\xi, \eta), \quad \phi_i(\xi, \eta) = F_i(\xi, \eta) - \sum_{j=1}^{i-1} \alpha_{ij}\phi_j(\xi, \eta), \tag{8}$$

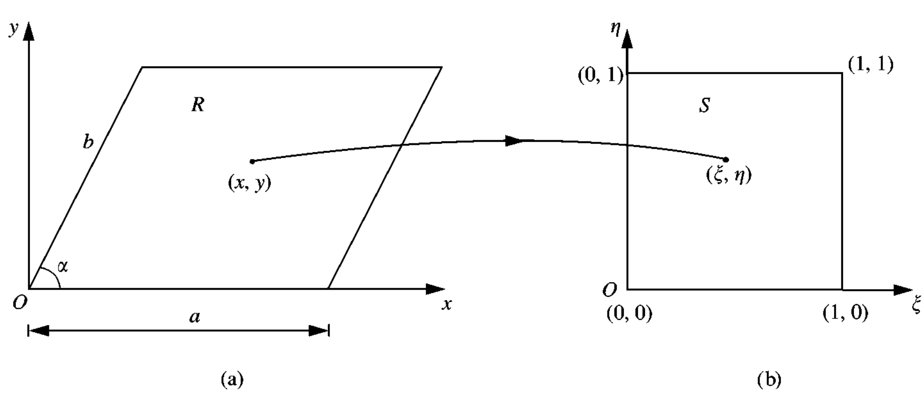

Figure 2. Mapping of the skew plate into a unit square.

where

$$\alpha_{ij} = \frac{\iint_S F_i(\xi,\eta)\phi_j(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{\iint_S \phi_j(\xi,\eta)\phi_j(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}, \quad j = 1,\ldots,i-1, \quad i = 2,3,4,\ldots,m. \tag{9}$$

After defining the orthogonal set of polynomials, the midsurface translational displacement in the z direction, $\varpi(\xi,\eta)$, can be assumed in the form of a finite series,

$$\varpi(\xi,\eta) = \sum_{i=1}^{m} C_i\phi_i(\xi,\eta), \tag{10}$$

where C_i's are undetermined coefficients. Substitution of $\varpi(\xi,\eta)$ into the mapped form of equation (4), and minimizing gives the following equation:

$$\sum_{j=1}^{m} [K_{ij} - \lambda^2 M_{ij}]C_j = 0, \quad i = 1,2,\ldots,m, \tag{11}$$

where K_{ij} is the stiffness matrix shown below:

$$K_{ij} = \frac{1}{s^4}\int_0^1\int_0^1 \Bigg\{ S_1 \frac{\partial^2\phi_i}{\partial\xi^2}\frac{\partial^2\phi_j}{\partial\xi^2} + S_2 \frac{\partial^2\phi_i}{\partial\eta^2}\frac{\partial^2\phi_j}{\partial\eta^2} + \frac{S_3}{2}\left(\frac{\partial^2\phi_i}{\partial\xi^2}\frac{\partial^2\phi_j}{\partial\eta^2} + \frac{\partial^2\phi_i}{\partial\eta^2}\frac{\partial^2\phi_j}{\partial\xi^2}\right)$$
$$+ S_4 \frac{\partial^2\phi_i}{\partial\xi\partial\eta}\frac{\partial^2\phi_j}{\partial\xi\partial\eta} + \frac{S_5}{2}\left(\frac{\partial^2\phi_i}{\partial\xi^2}\frac{\partial^2\phi_j}{\partial\xi\partial\eta} + \frac{\partial^2\phi_j}{\partial\xi^2}\frac{\partial^2\phi_i}{\partial\xi\partial\eta}\right)$$
$$+ \frac{S_6}{2}\left(\frac{\partial^2\phi_i}{\partial\eta^2}\frac{\partial^2\phi_j}{\partial\xi\partial\eta} + \frac{\partial^2\phi_j}{\partial\eta^2}\frac{\partial^2\phi_i}{\partial\xi\partial\eta}\right)\Bigg\}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{12}$$

S_i's are given in Appendix A, and M_{ij} is the mass matrix,

$$M_{ij} = \int_0^1\int_0^1 \phi_i\phi_j\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{13}$$

and λ^2 is the following frequency parameter:

$$\lambda^2 = \omega^2 a^4 \rho_0 / h^2 E_{22}. \tag{14}$$

2.2. BOUNDARY CONDITIONS

Plates clamped and simply supported at all sides are considered separately. The boundary conditions are as follows:

For a simply supported edge:

$$\xi = 0 \text{ and } 1: \quad \varpi = 0, \quad \partial^2\varpi/\partial\xi^2 = 0,$$
$$\eta = 0 \text{ and } 1: \quad \varpi = 0, \quad \partial^2\varpi/\partial\eta^2 = 0, \tag{15}$$

For a clamped edge:

$$\xi = 0 \text{ and } 1: \quad \varpi = 0, \quad \partial\varpi/\partial\xi = 0,$$
$$\eta = 0 \text{ and } 1: \quad \varpi = 0, \quad \partial\varpi/\partial\eta = 0. \tag{16}$$

3. RESULTS

Natural frequencies of skew plates are obtained for various types of material combinations and lamination schemes. Simply supported (SSSS) and clamped (CCCC) skew plates with various skew angles α are studied. Cross-ply and angle-ply laminates are used to examine the effect of the lay-up on frequencies. The results are given in the following non-dimensional frequency parameter form:

$$\lambda^* = \omega a^2/\pi^2 h\sqrt{\rho_0/E_{22}}. \tag{17}$$

Note that in equation (17) $\lambda^* = \lambda/\pi^2$. Convergence of λ^* values are given in Tables 1 and 2, for simply supported isotropic and cross-ply laminated plates.

First, a laminated skew plate with five symmetric cross-ply layers (90°/0°/90°/0°/90°) is analyzed. For such a symmetric lay-up there is no coupling between in-plane and out-of-plane behaviours. There is no bending–twisting coupling as a result of the cross-ply lay-up used. The results are presented in Tables 3 and 4 for the two different boundary conditions, SSSS and CCCC, respectively. The material properties of each lamina are identical and have the following values: $E_{11}/E_{22} = 40$, $G_{12}/E_{22} = 0{\cdot}6$, $v_{12} = 0{\cdot}25$.

In Table 4, the frequency results for a fully clamped skew plate are presented and they are in very good agreement with the results of Wang [20] for various skew angles and modes. The accuracy of results decreases as the skew angle decreases from 90°. When the results for a simply supported plate presented in Table 3 and the results for a fully clamped plate presented in Table 4 are compared, it is seen that the results for a fully clamped skew plate are closer to the results presented by Wang [20].

In Tables 5 and 6, the results for a laminated skew composite plate with five symmetric angle-ply (45°/ − 45°/45°/ −45°/45°) layers are presented for SSSS and CCCC boundary conditions respectively. In this case, as a result of the angle-ply configuration, there is

TABLE 1

Convergence of λ^* values for SSSS skew (α = 60°) isotropic plate

m	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*
9	2·59	6·77	9·25	12·48	19·37	19·97	22·18
16	2·58	5·45	7·52	9·49	18·72	18·83	19·81
20	2·57	5·40	7·46	8·89	13·23	13·81	14·76
25	2·54	5·36	7·39	8·68	13·07	13·41	14·59
29	2·54	5·34	7·33	8·54	12·77	13·20	14·59
30	2·57	5·29	7·19	8·64	12·81	13·19	14·63
36	2·54	5·33	7·31	8·53	12·74	12·75	14·54
40	2·54	5·33	7·31	8·51	12·51	12·52	14·31
49	2·54	5·33	7·30	8·50	12·48	12·45	14·26
Wang [20]	2·53	5·33	7·28	8·50	12·45	12·45	14·26
Singh *et al.* [13]	2·57	5·35	7·40	8·86	13·20	—	—

TABLE 2

Convergence of λ^* values for SSSS skew ($\alpha = 60°$) cross-ply [90°/0°/90°/0°/90°] laminated composite plate

m	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*
9	2·95	6·75	12·47	14·08	18·29	24·45	30·64
16	2·89	5·39	9·76	10·32	17·04	19·57	29·98
20	2·88	5·29	9·12	9·56	13·37	14·49	21·08
25	2·86	5·27	8·90	9·49	12·86	13·96	20·67
29	2·86	5·23	8·61	9·39	12·63	13·52	19·56
30	2·86	5·22	8·61	9·39	12·63	13·12	19·56
36	2·85	5·21	8·58	9·32	12·48	12·75	18·63
40	2·85	5·21	8·52	9·31	12·28	12·35	18·51
49	2·84	5·20	8·49	9·29	12·18	12·25	17·18
Wang [20]	2·83	5·19	8·48	9·26	12·11	12·13	16·48

TABLE 3

Values of λ^* for SSSS skew laminated [90°/0°/90°/0°/90°] composite plate

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	1·91	3·98	6·66	7·66	8·15	10·63	14·77	Present study
	1·91	3·98	6·66	7·66	8·15	10·63	14·19	Wang [20]
60°	2·84	5·20	8·49	9·29	12·18	12·25	17·18	Present study
	2·83	5·19	8·48	9·26	12·11	12·13	16·48	Wang [20]
45°	4·55	7·14	10·53	14·45	15·18	19·02	21·34	Present study
	4·48	7·11	10·45	14·10	14·78	17·96	19·60	Wang [20]

TABLE 4

Values of λ^* for CCCC skew laminated [90°/0°/90°/0°/90°] composite plate

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	4·24	6·69	10·45	11·44	11·78	15·14	18·29	Present study
	4·24	6·69	10·45	11·44	11·78	15·14	18·19	Wang [20]
60°	5·63	8·33	12·40	14·13	16·76	16·97	21·85	Present study
	5·63	8·33	12·40	14·13	16·74	16·96	21·70	Wang [20]
45°	8·46	11·80	16·18	21·09	21·56	25·98	27·48	Present study
	8·46	11·80	16·17	20·94	21·51	25·65	26·84	Wang [20]

a bending-twisting coupling. The material properties of each lamina are the same as the properties used previously. The results for angle-ply, simply supported laminated plate are in good agreement with the results of Wang [20] (see Table 5). For a fully clamped skew plate there is better accordance for higher modes and low skew angles as shown in Table 6.

TABLE 5

Values of λ for SSSS skew laminated* [45°/ − 45°/45°/ − 45°/45°] *composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	2·44	4·99	6·19	8·49	10·26	11·66	12·87	Present study
	2·43	4·99	6·18	8·49	10·25	11·65	12·83	Wang [20]
60°	2·61	5·69	6·84	9·48	11·90	13·25	14·35	Present study
	2·61	5·69	6·83	9·48	11·89	13·24	14·28	Wang [20]
45°	3·33	6·90	9·73	10·73	15·65	16·31	19·44	Present study
	3·32	6·90	9·69	10·72	15·53	16·15	19·35	Wang [20]

TABLE 6

Values of λ for CCCC skew laminated* [45°/ − 45°/45°/ − 45°/45°] *composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	3·90	7·15	8·46	11·21	13·32	14·75	16·13	Present study
	3·90	7·15	8·46	11·21	13·32	14·74	16·13	Wang [20]
60°	4·54	8·38	9·88	12·85	15·69	17·489	18·35	Present study
	4·54	8·38	9·88	12·85	15·69	17·49	18·34	Wang [20]
45°	6·31	10·82	14·50	15·47	21·09	22·13	25·89	Present study
	6·31	10·82	14·50	15·47	21·06	22·08	25·89	Wang [20]

TABLE 7

Values of λ for CCCC skew laminated* [30°/ − 30°/30°] *graphite/epoxy composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	2·47	3·77	5·70	6·03	7·56	8·03	10·17	Present study
	2·47	3·77	5·70	6·03	7·56	—	—	Hosokawa *et al.* [18]
	2·47	3·77	5·70	6·03	7·56	—	—	Han and Dickinson [19]
80°	2·31	3·82	5·42	5·96	7·28	8·49	10·00	Present study
	2·31	3·82	5·42	5·96	7·29	—	—	Hosokawa *et al.* [18]
	2·31	3·82	5·42	5·96	7·29	—	—	Han and Dickinson [19]
70°	2·27	4·09	5·04	6·52	7·40	9·12	9·39	Present study
	2·27	4·09	5·04	6·52	7·40	—	—	Hosokawa *et al.* [18]
	2·27	4·09	5·04	6·52	7·40	—	—	Han and Dickinson [19]
60°	2·37	4·65	4·95	7·35	8·15	8·76	10·87	Present study
	2·37	4·65	4·95	7·35	8·15	—	—	Hosokawa *et al.* [18]
	2·37	4·65	4·95	7·35	8·15	—	—	Han and Dickinson [19]
50°	2·71	5·22	5·74	8·14	9·46	10·11	11·88	Present study
	2·71	5·22	5·74	8·14	9·46	—	—	Hosokawa *et al.* [18]
	2·71	5·22	5·74	8·14	9·46	—	—	Han and Dickinson [19]

In Tables 7–9 non-dimensional frequencies of angle-ply rhombic laminates (30°/ − 30°/30°) are tabulated for skew, clamped laminated plates made of three types of composites; *graphite/epoxy*, *E-glass/epoxy* and *boron/epoxy*. The effect of the skew angle α on non-dimensional natural frequencies is calculated. The results are compared with those

TABLE 8

Values of λ^* for CCCC skew laminated [30°/ − 30°/30°] *E-glass/epoxy composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	1·32	2·46	2·90	3·88	4·40	5·28	5·58	Present study
	1·32	2·46	2·90	3·88	4·40	—	—	Hosokawa *et al.* [18]
80°	1·32	2·55	2·80	4·00	4·47	5·08	5·90	Present study
	1·32	2·55	2·80	4·00	4·47	—	—	Hosokawa *et al.* [18]
70°	1·39	2·69	2·94	4·14	4·89	5·23	6·05	Present study
	1·39	2·69	2·94	4·14	4·89	—	—	Hosokawa *et al.* [18]
60°	1·55	2·89	3·41	4·37	5·56	5·98	6·23	Present study
	1·55	2·89	3·41	4·37	5·56	—	—	Hosokawa *et al.* [18]
50°	1·88	3·30	4·29	4·85	6·66	6·72	7·60	Present study
	1·88	3·30	4·29	4·85	6·66	—	—	Hosokawa *et al.* [18]

TABLE 9

Values of λ^* for CCCC skew laminated [30°/ − 30°/30°] *boron/epoxy composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	2·10	3·27	5·02	5·07	6·42	7·13	8·72	Present study
	2·10	3·27	5·02	5·07	6·42	—	—	Hosokawa *et al.* [18]
80°	1·97	3·33	4·57	5·27	6·22	7·58	8·39	Present study
	1·97	3·33	4·57	5·26	6·22	—	—	Hosokawa *et al.* [18]
70°	1·95	3·58	4·28	5·80	6·34	7·67	8·44	Present study
	1·95	3·58	4·28	5·80	6·34	—	—	Hosokawa *et al.* [18]
60°	2·07	4·06	4·27	6·52	7·01	7·52	9·71	Present study
	2·07	4·06	4·27	6·52	7·01	—	—	Hosokawa *et al.* [18]
50°	2·39	4·53	5·08	7·08	8·37	8·83	10·25	Present study
	2·39	4·53	5·08	7·08	8·37	—	—	Hosokawa *et al.* [18]

of Hosokawa *et a*l. [18], and Han and Dickinson [19]. The following material properties are used for the graphite/epoxy composite: $E_{11}/E_{22} = 15{\cdot}4$, $G_{12}/E_{22} = 0{\cdot}79$, $v_{12} = 0{\cdot}3$. In Table 7, the results show that the natural frequency parameters are almost the same as the results of Hosokawa *et al.* [18], and Han and Dickinson [19] for graphite/epoxy. Hosokawa *et al.* [18] use the Green function with 66 number of terms. Han and Dickinson [19] use a hierarchical finite element method with 20 shape functions. The accuracy remains the same even for highly skewed plates and for higher modes.

E-glass/epoxy and boron/epoxy are the materials used to calculate the results of Tables 8 and 9. They have the following material properties, respectively: $E_{11}/E_{12} = 2{\cdot}45$, $G_{12}/E_{22} = 0{\cdot}48$, $v = 0{\cdot}23$, and $E_{11}/E_{22} = 11$, $G_{12}/E_{22} = 0{\cdot}34$, $v_{12} = 0{\cdot}21$.

The results are compared to the results of Hosokawa *et al.* [18] in Tables 8 and 9 for both cases. The frequencies agree quite well for highly skewed cases and for higher vibration modes. The results obtained in the present study for E-glass/epoxy are lower than those of Hosokawa *et al.* [18]. Graphite/epoxy yield the largest frequency values while

TABLE 10

Values of λ^* for CCCC skew laminated [0°/0°/0°/0°/0°] *graphite/epoxy composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	2·75	3·42	4·80	6·93	7·25	7·77	8·80	Present study
	2·75	3·42	4·80	6·90	7·25	—	—	Han and Dickinson [19]
30°	5·02	7·82	11·20	12·08	15·13	16·28	19·56	Present study
	5·02	7·81	11·20	12·07	15·08	—	—	Han and Dickinson [19]

TABLE 11

Values of λ^* for CCCC skew laminated [30°/ − 30°/30°/ − 30°/0°] *graphite/epoxy composite plate*

α	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	λ_7^*	Source
90°	2·62	4·21	6·22	6.58	8·07	9·60	10·97	Present study
	2·62	4·21	6·22	6·58	8·07	—	—	Han and Dickinson [19]
30°	6·06	9·92	13·81	14·46	18·41	21·14	23·60	Present study
	6·06	9·92	13·81	14·44	18·36	—	—	Han and Dickinson [19]

E-glass/epoxy material has the lowest frequency values for the symmetric angle-ply laminated composites.

Non-dimensional natural frequency parameters are also obtained for five symmetric angle-ply skew composite plates made of graphite/epoxy with (0°/0°/0°/0°/0°) and (30°/ − 30°/30°/ − 30°/30°) configurations. The material properties are the same as those used previously. For a fully clamped plate, the results are given in Tables 10 and 11 for two different skew angles ($\alpha = 90, 30$). There is very good agreement with the results of this study and of those of Han and Dickinson [19]. Although the laminate is highly skewed, the accuracy is very good even for higher modes.

4. DISCUSSION

In this paper, the free vibration problem of skew laminated composite plates is studied. Natural frequencies are calculated for various skew angles, lamination schemes and material properties. An approach that can easily be applied to finding the frequencies of laminated, skew, composite plates is provided.

The skew domain is mapped into a unit square to use simple orthogonal polynomials, which are generated through the Gram–Schmidt orthogonalization procedure. The use of orthogonal polynomials together with the Ritz method is an efficient computational method. Once the orthogonal polynomials are generated for the unit square, they are used for plates with different aspect ratios a/b, and skew angles α. An appropriate number of polynomials, m can be chosen to obtain a desired accuracy. In this study, 49 terms have been used for the calculations to obtain sufficient accuracy. The next step would be to use 64 terms, which would introduce additional computational difficulties, and make the computation time considerably longer. Mathematica has been used extensively, and for example the use of 49 terms instead of 36 almost doubled the amount of computation time.

For all cases, 49 terms are used, and 49 terms give satisfactory results in frequency parameters when compared with the previous studies. The comparison of frequency parameters is given in Tables 3–11. For example, Liew and Lam [11] use 35 terms and the convergence pattern they show for the first four modes employs two decimal places. A similar convergence pattern for the first four modes has been obtained as seen in Tables 1 and 2.

The frequency parameters are evaluated for simply supported and clamped laminated composite skew plates. The accuracy obtained in frequency parameters for different modes and skew angles using 49 terms is presented in Tables 3–11. The results show that sufficient accuracy has been obtained when the results are compared to those of Wang [20], Han and Dickinson [19] and Hosokawa *et al.* [18]. Higher accuracy in the results is obtained for the case of clamped plates. This may be the result of the displacement form assumed for the simply supported case. The accuracy is determined by comparision with the results presented in the literature.

In frequency parameter calculations, the accuracy of the results decreases as the number of modes increases. To increase the accuracy for higher modes more terms should be included to the series expression for $\varpi(\xi, \eta)$, given in equation (10). Accuracy also decreases for the cases where α decreases from 90°.

We have used simple polynomials that are easy to apply compared to other methods used in the literature such as the use of B-spline function by Wang [20], Green function by Hosokawa *et al.* [18]. This makes the application of the solution method to skew plates less tedious.

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APPENDIX A

$$D_{ij} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \bar{Q}_{ij}^{(k)} z^2 \, dz, \tag{A1}$$

where the $\bar{Q}_{ij}$'s $(i, j = 1, 2, 6)$ are

$$\bar{Q}_{11}^{(k)} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta,$$

$$\bar{Q}_{11}^{(k)} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta,$$

$$\bar{Q}_{12}^{(k)} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}((\sin^4\theta + \cos^4\theta),$$

$$\bar{Q}_{66}^{(k)} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta),$$

$$\bar{Q}_{16}^{(k)} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta,$$

$$\bar{Q}_{16}^{(k)} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \tag{A2}$$

and the elastic constants Q_{ij}'s $(i, j = 1, 2, 6)$ are

$$Q_{11} = E_{11}/(1 - v_{12}v_{21}), \quad Q_{22} = E_{22}/(1 - v_{12}v_{21}),$$

$$Q_{12} = v_{21}E_{11}/(1 - v_{12}v_{21}), \quad Q_{21} = v_{12}E_{22}/(1 - v_{12}v_{21}),$$

$$Q_{66} = G_{12}, \quad Q_{16} = Q_{26} = 0, \tag{A3}$$

$$S_1 = D^*_{11} + \frac{2c^2 D^*_{12}}{s^2} + \frac{c^4 D^*_{22}}{s^4} + \frac{4c^2 D^*_{66}}{s^2} - \frac{4cD^*_{16}}{s} - \frac{4c^3 D^*_{26}}{s^3},$$

$$S_2 = \frac{\mu^4 D^*_{22}}{s^4},$$

$$S_3 = \frac{2\mu^2 D^*_{12}}{s^2} + \frac{2\mu^2 c^2 D^*_{22}}{s^4} - \frac{4\mu^2 c D^*_{26}}{s^3},$$

$$S_4 = \frac{4\mu^2 c^2 D^*_{22}}{s^4} - \frac{8\mu^2 c D^*_{26}}{s^3} + \frac{4\mu^2 D^*_{66}}{s^2},$$

$$S_5 = -\frac{4\mu c D^*_{12}}{s^2} + \frac{4\mu D^*_{16}}{s} - \frac{4\mu c^3 D^*_{22}}{s^4} + \frac{12\mu c^2 D^*_{26}}{s^3} - \frac{8\mu c D^*_{66}}{s^2},$$

$$S_6 = -\frac{4\mu^3 c D^*_{22}}{s^4} + \frac{4\mu^3 D^*_{26}}{s^3}, \tag{A4}$$

$$D^*_{ij} = D_{ij}/E_{22}h^3, \tag{A5}$$

$$\mu = a/b, \tag{A6}$$

where $s = \sin\alpha$ and $c = \cos\alpha$.

Displacement function used for simply supported boundary:

$$\begin{aligned}\varpi(\xi,\eta) = {} & C_1(\xi\eta - \xi^2\eta - \xi\eta^2 + \xi^2\eta^2) + C_2(-\xi\eta/2 + 3\xi^2\eta/2 - \xi^3\eta + \xi\eta^2/2 - 3\xi^2\eta^2/2 \\ & + \xi^3\eta^2) + C_3(-\xi\eta/2 + \xi^2\eta/2 + 3\xi\eta^2/2 - 3\xi^2\eta^2/2 - \xi\eta^2 + \xi^2\eta^3) \\ & + C_4(\xi\eta/4 - 3\xi^2\eta/4 + \xi^3\eta/2 - 3\xi\eta^2/4 + 9\xi^2\eta^2/4 - 3\xi^3\eta^2/2 + \xi\eta^3/2 \\ & - 3\xi^2\eta^3/2 + \xi^3\eta^3).\end{aligned} \tag{A7}$$

Displacement function for CCCC boundary conditions:

$$\begin{aligned}\varpi(\xi,\eta) = {} & C_1(\xi^2\eta^2 - 2\xi^3\eta^2 + \xi^4\eta^2 - 2\xi^2\eta^3 + 4\xi^3\eta^3 - 2\xi^4\eta^3 + \xi^2\eta^4 - 2\xi^3\eta^4 + \xi^4\eta^4) \\ & + C_2(-\xi^2\eta^2/2 + 2\xi^3\eta^2 - 5\xi^4\eta^2/2 + \xi^5\eta^2 + \xi^2\eta^3 - 4\xi^3\eta^3 + 5\xi^4\eta^3 \\ & - 2\xi^5\eta^3 - \xi^2\eta^4/2 + 2\xi^3\eta^4 - 5\xi^4\eta^4/2 + \xi^5\eta^4) + C_3(-\xi^2\eta^2/2 + \xi^3\eta^2 \\ & - \xi^4\eta^2/2 + 2\xi^2\eta^3 - 4\xi^3\eta^3 + 2\xi^4\eta^3 - 5\xi^2\eta^4/2 + 5\xi^3\eta^4 - 5\xi^4\eta^4/2 \\ & + \xi^2\eta^5 - 2\xi^3\eta^5 + \xi^4\eta^5) + C_4(\xi^2\eta^2/4 - \xi^3\eta^2 + 5\xi^4\eta^4/4 - \xi^5\eta^2/2 - \xi^2\eta^3 \\ & + 4\xi^3\eta^3 - 5\xi^4\eta^3 + 2\xi^5\eta^3 + 5\xi^2\eta^4/4 - 5\xi^3\eta^4 + 25\xi^4\eta^4/4 - 5\xi^5\eta^4/2 \\ & - \xi^2\eta^5/2 + 2\xi^3\eta^5 - 5\xi^4\eta^5/2 + \xi^5\eta^5).\end{aligned} \tag{A8}$$